Fluctuation of the strength function

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(Received 14 May 2001; revised manuscript received 3 April 2002; published 22 August 2002)

We consider a single state stochastically coupled to its stochastic background states. The fluctuation of the strength function of the single state is systematically studied. We find that the upper and lower deviations of the strength function only depend on the ratio of the spreading width over the decay width of the single state and on the ratio of the common decay width over the mean level spacing of the background states. Based on the two fit formulas for the upper and lower deviations, the uncertainties of the full width at half maximum (FWHM) and lifetime of a single state are estimated. They predict the experimental error bars of the FWHM and lifetime. A comparison of the uncertainties with the experimental error bars is made for nuclear giant dipole resonance, which illuminates our theoretical predictions.

DOI: 10.1103/PhysRevE.66.026208 PACS number(s): 05.45.Mt, 24.60.Lz

The mixing of a single state with its complicated background states due to residual interactions or external perturbations is ubiquitous and important in a variety of fields ranging from condensed matter to atomic nuclei. The single state can be, for example, a collective mode in many-body systems. The mixing makes the single state spread out over its background states and acquire a spreading width that in most cases dominates the lifetime of the single state. Using the language of time evolution the mixing causes the damping of the single state. Such mixing can be conveniently described by the strength function that was introduced for the first time by Wigner in 1950s [1] and defined as

$$S(E)=\sum_j |\langle 0|j\rangle|^2 \delta(E-E_j). \tag{1}$$

Here $|0\rangle$ denotes the single state and $|j\rangle$ and E_j are the eigenstates and eigenvalues of the total Hamiltonian of the coupled systems. In the 1960s the strength function was well discussed in the context of nuclear physics by Bohr and Mottelson [2].

Recently the strength function has received growing attention. It has been applied to atomic clusters [3] the rare-earth atom of Ce [4], nuclear structure [5,6]. It has been also studied in the context of the band-random matrix and embedded random matrix theories [7,8] as well as the Lipkin model [9] and the two-dimensional anharmonic oscillator model [10].

The complexity of the background states and their couplings with the single state justifies statistical treatments in which the strength function appears as a statistical quantity. The average strength function has been extensively investigated. It has been recognized that usually the average strength function follows the Breit-Wigner (BW) formula and the deviations from the BW formula can be found in some cases [4,5,7]. A statistical quantity should be characterized by its average as well as its fluctuation. To the best of our knowledge, however, nobody has discussed the fluctuation of the strength function.

In the present paper we consider a single state coupling with its complicated background states. Being complicated the background states and their couplings with the single state are treated stochastically. In many physical situations, for example, in nuclear giant dipole resonances (GDR) and the decay out of a superdeformed band [11] both the single and background states are no longer pure stationary states. They are unstable against particle or γ emission. Therefore, as a general consideration, we shall introduce the decay widths for the single and background states. The system then becomes open and its Hamiltonian non-Hermitian. Such a kind of Hamiltonian was used by Humblet and Rosenfeld [12] and Mahaux and Weidenmüller [13] on nuclear reactions in 1960s, and later by Sokolov and Zelevinsky on a statistical theory of overlapping resonances [14] and by Weidenmüller *et al.* on the decay out of a superdeformed band [15]. We shall focus on the fluctuation of the strength function, full width at half maximum (FWHM) and lifetime of the single state.

To define our model we denote the single state by $|0\rangle$; its energy by E_0. The background states are modeled as eigenstates of the Gaussian orthogonal ensemble (GOE) of random matrices [11,15,16]. We denote them by $|j\rangle$ with $j=1,\ldots,K$ and $K\gg 1$; their energies by E_j. We denote the decay width of the single state by Γ_0. We assume that the decay widths of all the background states have the common value Γ_b. The matrix elements V_{0j} connecting the single and the background states are responsible for their mixing. The energies E_j follow the GOE distribution, and the V_{0j}'s are uncorrelated Gaussian distributed random variables with mean value zero and common variance v^2. The spreading width $\Gamma^{\downarrow}$ is defined by Fermi golden rule $\Gamma^{\downarrow}=2\pi v^2/d$, with d the mean level spacing of the background states.

The Hamiltonian H of the system is a matrix of dimension $K+1$ and has the form $(j,l=1,\ldots,K)$

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$$H=\begin{pmatrix} E_0 & V_{0j} \\ V_{0l} & \delta_{jl}E_j \end{pmatrix}. \tag{2}$$

To H must be added the diagonal width matrix Σ_{0B} given by

$$\Sigma_{0B}=-(i/2)\begin{pmatrix} \Gamma_0 & 0 \\ 0 & \delta_{jl}\Gamma_b \end{pmatrix}. \tag{3}$$

The effective Hamiltonian H_{eff} is given by

$$H_{\text{eff}}=H+\Sigma_{0B}. \tag{4}$$

The strength function can be expressed in terms of the resolvent of the effective Hamiltonian

$$S(E)=-\frac{1}{\pi}\text{Im}\left(\left\langle 0\left|\frac{1}{E-H_{\text{eff}}}\right|0\right\rangle\right). \tag{5}$$

Equation (5) is identical to Eq. (1) for vanishing decay widths. $S(E)$ varies with the realization of the ensemble of random matrices. Its ensemble average involves an average over both, the distribution of matrix elements V_{0j} and the distribution of eigenvalues E_j. By means of the supersymmetry approach, the ensemble average of $S(E)$ can be worked out analytically [15,17] and denoted by a bar,

$$\overline{S(E)}=\frac{1}{2\pi}\frac{\Gamma_0+\Gamma^{\downarrow}}{(E-E_0)^2+(\Gamma_0+\Gamma^{\downarrow})^2/4}. \tag{6}$$

It exactly follows the BW formula and is independent of Γ_b. The limit $K\to\infty$ has been taken in calculation.

The calculation of the fluctuation of $S(E)$ around its average is beyond the scope of the supersymmetry technique. Therefore we employ numerical simulation to compute the fluctuation. In our numerical calculations we use matrices of dimension $K=1000$ or bigger and calculate $N=10^4$ or more realizations. As a result the calculated average of $S(E)$ coincides with that given by formula (6). This constitutes a test for the accuracy of our numerical simulation. For convenience Γ_0, Γ_b, $\Gamma^{\downarrow}$, E, and E_0 are measured in units of the mean level spacing d and d is taken to be a unit. Therefore, they are dimensionless, and their values indicate the ratios of them to d.

With $S(E)^{(n)}$ the value of $S(E)$ obtained in the nth realization $(n=1,\ldots,N)$, two sets labeled with $i=1,2$ are formed depending on whether $S(E)^{(n)}>\overline{S(E)}$ or $S(E)^{(n)}<\overline{S(E)}$, respectively, each set containing N_i realizations labeled $\mu_i=1,\ldots,N_i$. We then define the standard upper deviation $\sigma^{(1)}(E)$ and lower deviation $\sigma^{(2)}(E)$,

$$\sigma^{(i)}(E)=\sqrt{\frac{1}{N_i}\sum_{\mu_i}^{N_i}[S(E)^{(\mu_i)}-\overline{S(E)}]^2}. \tag{7}$$

Here we would emphasize that the fluctuation involved in random matrix models is qualitatively different from that in the statistical mechanics of macroscopic bodies [18]. The former results from the stochasticity of the relevant random matrix and random couplings. It is inherent and will never vanish as the dimension of the matrix goes to infinity. It will approach constants that are independent of the dimension if the dimension is large enough. The latter is usually the result of a coupling to a heat bath. It will vanish in the thermodynamical limit. We notice that the upper and lower deviations become constants independent of the dimension and number of realizations if the dimension and number are large enough. We show how the upper and lower deviations change with $\Gamma^{\downarrow}$ in Figs. 1(a)–1(c) and with Γ_b in Figs. 1(d)–1(f). $\Gamma^{\downarrow}$ measures the stochastic coupling strength. When $\Gamma^{\downarrow}$ is small the single state spreads out over a few background states. Then the average strength function as well as its deviations are located within a small range of energy. This situation is illustrated in Fig. 1(a) where $\Gamma^{\downarrow}=0.05$. With the increase of $\Gamma^{\downarrow}$ the single state spreads out over more and more background states and the energy range where the deviations distribute becomes larger and larger as shown in Figs. 1(a)–1(c). At the same time from Figs. 1(a)–1(c) one can find that with increasing $\Gamma^{\downarrow}$ the deviations decrease in magnitude, however, their ratios to the average strength function keep going up. The deviations strongly depend on Γ_b as illustrated in Figs. 1(d)–1(f). In the case of small Γ_b the deviations are very large. The deviations are reduced with the increase of Γ_b. Such behavior reflects directly the role played by individual background states. To understand this we show in Fig. 2 the strength function for a single realization as a function of energy for different values of Γ_b. In Fig. 2(a) Γ_b is small.

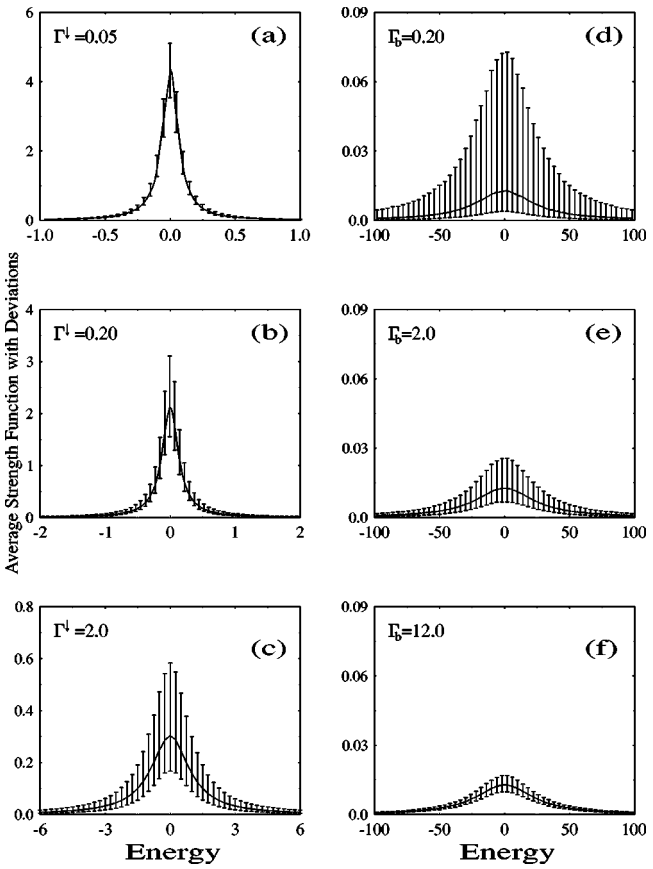

FIG. 1. Ensemble average of the strength function and its upper and lower deviations versus energy for different values of $\Gamma^{\downarrow}$ and Γ_b with $\Gamma_0=0.10$. (a)–(c) $\Gamma_b=2.0$, (d)–(f) $\Gamma^{\downarrow}=50.0$.

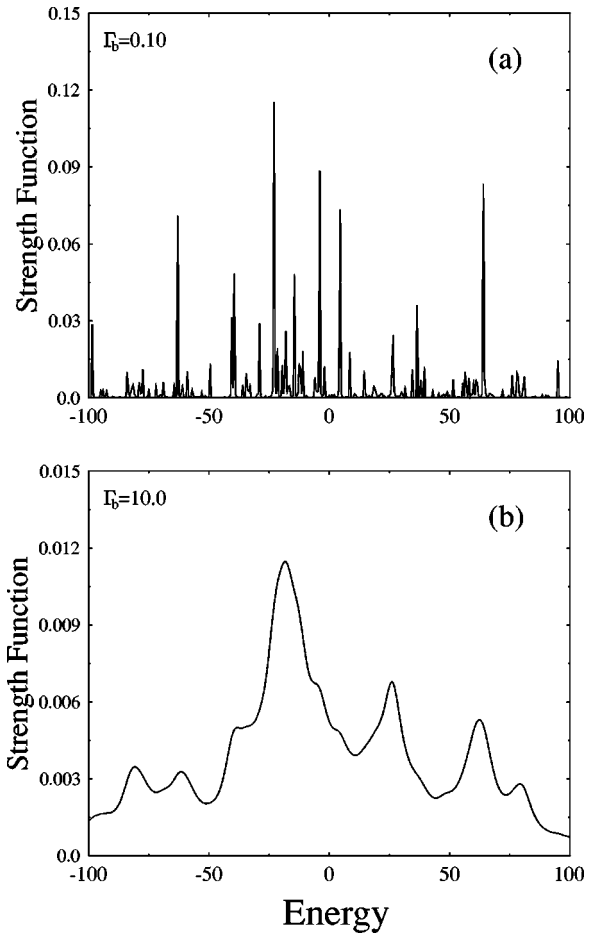

FIG. 2. Strength function versus energy in a single realization for different values of Γ_b with $\Gamma_0=0.10$ and $\Gamma^{\downarrow}=100.0$.

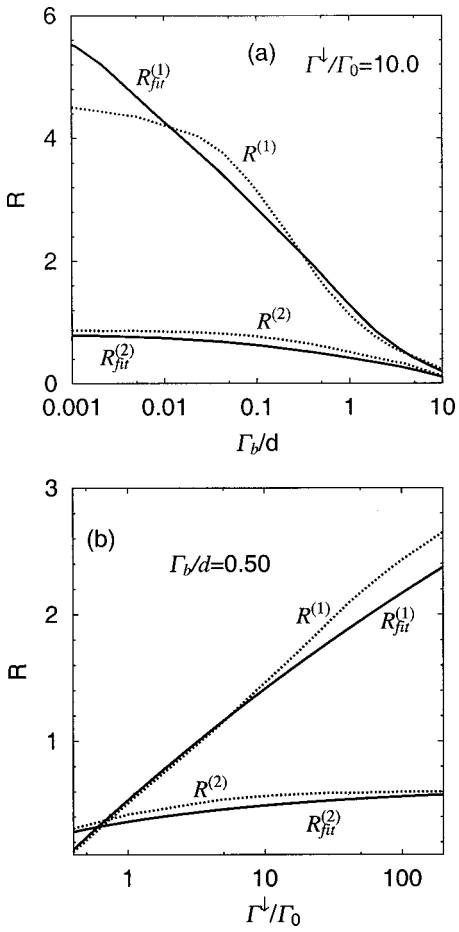

FIG. 3. Comparison of the ratios given by the fit formulas with those by the numerical simulations. Ratios are plotted as a function of Γ_b/d (a) and as a function of $\Gamma^{\downarrow}/\Gamma_0$ (b). The dashed lines show the numerical results.

Then the background states are well isolated and their positions strongly influence the locations of the peaks of $S(E)$. So in this case large deviations can be expected. When Γ_b is large the background states overlap strongly and their locations are less important to $S(E)$. Then $S(E)$ becomes smooth [see Fig. 2(b)] and small deviations prevail when Γ_b is large.

We find that the ratios of the deviations to the average keep constant for fixed values of $\Gamma^{\downarrow}$, Γ_0, and Γ_b, namely, they are independent of energy but depend on $\Gamma^{\downarrow}$, Γ_0, and Γ_b. Moreover, we find that the ratios only depend on $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d (as mentioned above $d=1$) as long as $\Gamma^{\downarrow}$ is not much small compared to Γ_0,

$$R^{(i)}(\Gamma^{\downarrow}/\Gamma_0,\Gamma_b/d)=\frac{\sigma^{(i)}(E)}{\overline{S(E)}} \quad (i=1,2). \tag{8}$$

Since $\overline{S(E)}$ is known, the ratios $R^{(i)}(\Gamma^{\downarrow}/\Gamma_0,\Gamma_b/d)$ well reflect the upper and lower deviations. Meanwhile this energy independence makes it possible to determine the dependence of the ratios on $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d. The ratios have been calculated when $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d are changed systematically. This allows us to obtain two fit formulas of the ratios as functions of $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d:

$$\begin{aligned} R^{(1)}_{fit}(\Gamma^{\downarrow}/\Gamma_0,\Gamma_b/d) &= 5(\Gamma_b/d)^{-0.45}[1-\{1-0.0927(\Gamma_b/d)^{0.245}\} \\ &\quad \times(\Gamma^{\downarrow}/\Gamma_0)^{[-0.001-0.0775(\Gamma_b/d)^{0.267}]}], \end{aligned} \tag{9}$$

$$\begin{aligned} R^{(2)}_{fit}(\Gamma^{\downarrow}/\Gamma_0,\Gamma_b/d) &= [1-0.461(\Gamma_b/d)^{0.304}e^{-0.165log^2(\Gamma_b/d)}] \\ &\quad \times e^{-0.001(\Gamma_b/d)}[1-0.628e^{-0.369(\Gamma_b/d)^{-0.0586}} \\ &\quad \times(\Gamma^{\downarrow}/\Gamma_0)^{[-0.147-0.126(\Gamma_b/d)^{-0.0570}]} \\ &\quad \times\exp[-0.286\{1 \\ &\quad -0.817(\Gamma_b/d)^{-0.0297}\}\log^2(\Gamma^{\downarrow}/\Gamma_0)]]. \end{aligned} \tag{10}$$

We emphasize that the formulas are not based on any theoretical arguments and present the result of an approach based on trial and error. We show a comparison between the fit formulas (solid lines) and the calculated values (dashed lines) in Fig. 3. Because both $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d range from very small to very large values it is a tough task to get fit formulas with a simple form and high precision. For $R^{(1)}$ the

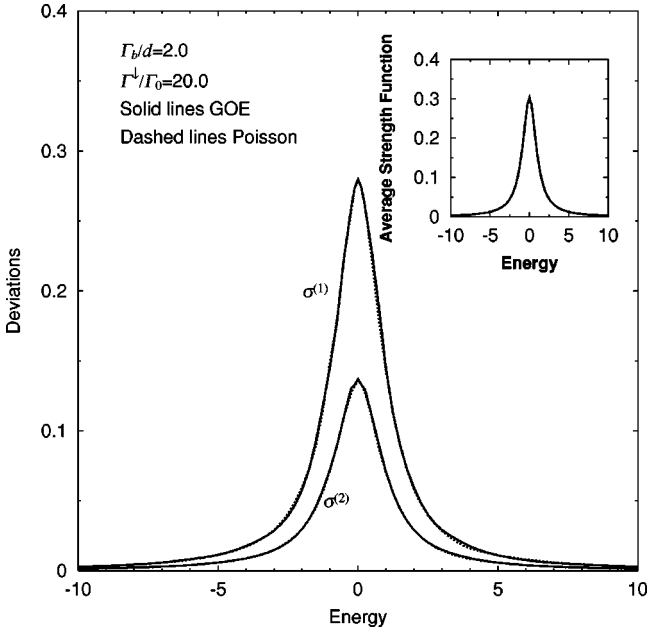

FIG. 4. Comparison of the GOE ensemble average (inset) of strength function and its deviations versus energy with those of the Poisson.

difference between the fit formulas and the calculated values becomes large if Γ_b/d is very small [see Fig. 3(a)]. However, it is less than 20% if Γ_b/d is larger than 10^{-3}. The two formulas tell us that the deviations grow with the increase of $\Gamma^{\downarrow}/\Gamma_0$ and are close to their saturating values when $\Gamma^{\downarrow}/\Gamma_0$ is large enough (up to several orders of magnitude). The deviations are large when Γ_b/d is much less than 1 (the background states are well isolated) and fall down with the increase of Γ_b/d. They become small when Γ_b and d are comparable and approach zero when Γ_b/d is large enough (the background states are well overlapped).

It is important to note that when the coupling matrix elements are kept and the background states are replaced by eigenstates of the Poisson ensemble of random matrices or by ones generated by the ensemble in between the Poisson and GOE [19] both the average and the deviations of the strength function are almost the same as those of the GOE. In Fig. 4 we plot the average (inset) and deviations as a function of energy for the cases of the GOE and Poisson distributions. One can see that the differences between the two cases in the average and deviations are difficult to be discerned. This suggests that the average and deviations depend on the stochasticity of the background states and couplings but are almost independent of the correlation and chaoticity of the background states. In the case of the GOE there is a strong level correlation, and the spectrum is chaotic. Nevertheless there is no level correlation, and the spectrum is regular in the case of the Poisson. It is implied that as far as the average in Eq. (6) and deviations in Eqs. (9) and (10) are concerned, the stochasticity of the background states is necessary but the chaoticity may not be necessary. Therefore the two fit formulas establish a general relationship between the fluctuation (deviations) of the strength function and the properties of the stochastic background states and their stochastic couplings with a single state.

The FWHM of the strength function can be evaluated in terms of the BW formula. It is inversely proportional to the peak value of the strength function. With the maximum (minimum) peak value obtained by adding (subtracting) the upper (lower) deviation to (from) the average peak value we can estimate the upper and lower uncertainties of the FWHM according to the inverse proportionality,

$$\Gamma_{full}^{upper}=\overline{\Gamma}_{full}\frac{R^{(2)}}{1-R^{(2)}}, \tag{11}$$

$$\Gamma_{full}^{lower}=\overline{\Gamma}_{full}\frac{R^{(1)}}{1+R^{(1)}}. \tag{12}$$

Here $\overline{\Gamma}_{full}=\Gamma_0+\Gamma^{\downarrow}$ is the mean FWHM. Further the lifetime uncertainties can be evaluated as

$$\tau^{upper}=\frac{\hbar}{\overline{\Gamma}_{full}}R^{(1)}, \tag{13}$$

$$\tau^{lower}=\frac{\hbar}{\overline{\Gamma}_{full}}R^{(2)}. \tag{14}$$

Obviously the uncertainties of the FWHM and lifetime go up as the ratios $R^{(i)}$ increase. This implies that these uncertainties increase with $\Gamma^{\downarrow}/\Gamma_0$ and decrease with Γ_b/d.

As was suggested in Ref. [18] the fluctuations observed in complex quantum many-body systems are the manifestation of their inherent chaotic dynamics. The chaotic dynamics is responsible for the stochasticity of the background states and couplings and physically justifies their stochastic description as well. A single realization of a random matrix ensemble corresponds to an experimental event and a random matrix ensemble to a set of experimental events. The ensemble average and deviations correspond to the experimental statistical average and statistical error bars which stem from the chaotic dynamics. Therefore formulas (9)–(14) predict the experimental statistical error bars of the FWHM and lifetime and provide important information for the relevant experimentalists. To exemplify our ideas we apply these formulas to calculate the deviations and uncertainties for nuclear GDR and compare them with the experimental data. As is well known nuclear GDR has been found in many nuclei throughout the periodic table and extensively studied both theoretically and experimentally. We take ^{208}Pb as an example to illustrate our calculation. In the case of nuclear GDR the single state is the GDR (a coherent superposition of one-particle one-hole states) and the background states are the many-particle many-hole states. For the calculation of the decay widths we use the computer code developed by Sheldon and Rogers [20]. It contains a global optical-model potential to compute the transmission coefficients of nucleons. These in turn are used to determine the mean absorption cross section and the decay widths of many-particle many-hole states using the exciton model [21]. The decay width for the GDR Γ_0 and the common decay width for the back-

ground states Γ_b are evaluated at the centroid energy of the GDR (13.5 MeV) for one-particle one-hole states and two-particle two-hole states, respectively. Using the method described above we obtain $\Gamma_0 = 0.11$ MeV and $\Gamma_b = 0.30$ MeV. The level mean spacing of the background states at the centroid energy is calculated using the level density of Fermi gas model, and is found to be 0.0027 eV. The spreading width $\Gamma^{\downarrow}$ is the difference between the FWHM (4.0 MeV) and decay width of the GDR and equal to 3.89 MeV. The FWHM is determined experimentally [22]. One can see that the spreading width dominates the FWHM and hence the lifetime. With the values of the decay widths, mean level spacing and spreading width, we have $\Gamma^{\downarrow}/\Gamma_0 = 35$ and $\Gamma_b/d = 1.1 \times 10^8$. By means of formulas (9) and (10), both the upper and lower deviations are found to be vanishing. Then, the uncertainties of the FWHM and lifetime are also vanishing according to formulas (11)–(14). The relevant experimental data are available in Ref. [23]. One can find that the experimental error bar of cross section of the GDR (which is obtained by integrating the strength function over impact parameter) is no more than 5%. This error bar might not be of statistics but of systematics. The error bars of the FWHM and lifetime are so small that they are neglected at all. So the theoretical predictions agree with the experimental data. For other nuclei, the conclusion remains the same.

The fluctuations in the present paper are calculated when the dimension of the matrix is large enough. They are suitable to describe the situations where the number of the background states is large enough, for instance, nuclear giant resonances. If the number of the background states becomes finite then the strength function and its fluctuation differ from those with infinite number [24]. We shall study the problem of finite background states in future.

In conclusion, we have found that the ratios of upper and lower deviations of the strength function to the average keep constant and only depend on $\Gamma^{\downarrow}/\Gamma_0$ and Γ_b/d if $\Gamma^{\downarrow}$ is not much smaller than Γ_0. The ratios are fitted by two formulas that establish a general relationship between the fluctuation (deviations) of the strength function and the properties of the stochastic background states and their stochastic couplings with a single state. Based on the two fit formulas we have estimated the uncertainties of the FWHM and lifetime of a single state that predict the corresponding experimental error bars. We have exemplified our theoretical predictions by comparing them with the experimental data of nuclear GDR and found an agreement between the theory and experiment. Formulas (9)–(14) shed some light on the understanding of fluctuation of the strength function, FWHM and lifetime of a single state mixed by its complex background states, for example, a collective mode in many-body systems.

J.G. is grateful to K. Matsuyanagi, Y. Z. Zhuo, H. Aiba, S. Mizutori, and K. Hagino for valuable discussions and to K. Matsuyanagi for helpful suggestions. This work was supported in part by the Hong Kong Research Grant Council and Hong Kong Baptist University Faculty Research Grant Program.

[1] E. Wigner, Ann. Math. **62**, 548 (1955).
[2] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1968), Vol. 1.
[3] K. Yabana and G.F. Bertsch, Phys. Rev. A **54**, 4484 (1996); *ibid.* **60**, 3809 (1999).
[4] V.V. Flambaum *et al.*, Phys. Rev. A **50**, 267 (1994).
[5] N. Frazier, B.A. Brown, and V. Zelevinsky, Phys. Rev. C **54**, 1665 (1996).
[6] M. Matsuo *et al.*, Phys. Lett. B **465**, 1 (1999).
[7] Y.V. Fyodorov *et al.*, Phys. Rev. Lett. **76**, 1603 (1996).
[8] V.K.B. Kota and R. Sahu, Phys. Rev. E **62**, 3568 (2000).
[9] W. Wang, F.M. Izrailev, and G. Casati, Phys. Rev. E **57**, 323 (1998).
[10] H. Aiba, S. Mizutori, and T. Suzuki, Phys. Rev. E **56**, 119 (1997); H. Aiba and M. Matsuo, *ibid.* **60**, 034307 (1999).
[11] J.Z. Gu and H.A. Weidenmüller, Nucl. Phys. A **660**, 197 (1999); *ibid.* **690**, 382 (2001).
[12] J. Humblet and L. Rosenfeld, Nucl. Phys. **26**, 529 (1961).
[13] C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).
[14] V.V. Sokolov and V.G. Zelevinsky, Phys. Lett. B **202**, 10 (1988).
[15] H.A. Weidenmüller, P. von Brentano, and B.R. Barrett, Phys. Rev. Lett. **81**, 3603 (1998).
[16] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[17] J.J.M. Verbaarschot, H.A. Weidenmüller, and M.R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
[18] O. Bohigas and H.A. Weidenmüller, Annu. Rev. Nucl. Part. Sci. **38**, 421 (1988).
[19] P. Persson and S. Åberg, Phys. Rev. E **52**, 148 (1995).
[20] E. Sheldon and V.C. Rogers, Comput. Phys. Commun. **6**, 99 (1973).
[21] G. Mantzouranis, H.A. Weidenmüller, and D. Agassi, Z. Phys. A **276**, 145 (1976).
[22] G.F. Bertsch, P.F. Bortignon, and R.A. Broglia, Rev. Mod. Phys. **55**, 287 (1983).
[23] B.L. Berman and S.C. Fultz, Rev. Mod. Phys. **47**, 713 (1975).
[24] T. Guhr and A. Müller-Groeling, J. Math. Phys. **38**, 1870 (1997).